STATIONARY HEAT CONDUCTION IN CYLINDER WITH PIECEWISE-CONSTANT COEFFICIENT OF HEAT EXCHANGE ON LATERAL SURFACE

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The reduction to infinite systems is extended to the case of boundary conditions of the third kind. A particular case of a portion of the lateral surface of a cylinder being thermally insulated is investigated.

In actual reinforcement design when designing nuclear reactors or in the computation of thermal states of the elements of radioelectronic devices one is often compelled to solve analytically stationary heat-conduction problems for bodies of complex geometry but which can be represented as a sum of two or several sufficiently simple domains. It appears that a suitable method for solving such problems is to reduce them to an infinite system of equations [1]. A similar method can also be developed for the case of boundary conditions of the third kind.

One considers as an example stationary heat conduction in a cylinder of finite height (see Fig. 1) on whose lateral surface convective heat exchange takes place with the temperature of the medium, T_{med}:

$$\frac{\partial\theta}{\partial r} + h(z)\,\theta|_{r=R} = 0, \tag{1}$$

where

$$h(z) = \begin{cases} h_1 = \text{const}, & z < a, \\ h_2 = \text{const}, & z > a. \end{cases}$$

It is assumed that the temperature of the bottom base is constant and equal to T_0, and that at the top base there takes place convective heat exchange:

$$\frac{\partial\theta}{\partial z} + h_2\theta|_{z=b} = 0. \tag{2}$$

The cylinder is subdivided into two smaller ones (I) and (II), and one sets

$$\theta = \begin{cases} \theta_1(r, z) & \text{in domain } (I) \\ \theta_2(r, z) & \text{in domain } (II) \end{cases}. \tag{3}$$

Then the functions $\theta_i (i = 1, 2)$ satisfy the Laplace equation $\Delta\theta_i = 0$ in the respective domains, as well as the boundary conditions (1)-(2).

Using the eigenfunctions method [2] the solution θ_2 is sought in the form

$$\theta_2(r, z) = \sum_{k=1}^{\infty} D_k \left[\mu_k \operatorname{ch}\mu_k \frac{b-z}{R} + h_2 R \operatorname{sh}\mu_k \frac{b-z}{R}\right] J_0\left(\mu_k \frac{r}{R}\right), \tag{4}$$

where μ_k are the positive roots of the characteristic equation

$$\mu J_1(\mu) = h_2 R J_0(\mu), \tag{5}$$

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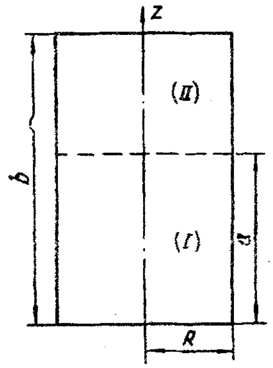

Fig. 1. Uniform cylinder with piecewise-constant coefficient of heat exchange on lateral surface.

and the condition (2), as well as the condition (1) of the third kind on the lateral surface r = R, are now satisfied.

The solution in the domain (I) which satisfies the condition of constant temperature at the bottom base and of continuous temperature at the section z = a is assumed to be a series of eigenfunctions of the corresponding homogeneous problem for the height coordinate z:

$$\theta_1(r,\ z)=\left(1-\frac{z}{a}\right)+\frac{2h_1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,I_0\left(\frac{n\pi}{a}\,r\right)\times\sin\frac{n\pi}{a}\,z\left[\frac{n\pi}{a}\,I_1\left(\frac{n\pi}{a}\,R\right)+h_1I_0\left(\frac{n\pi}{a}\,R\right)\right]^{-1}+$$

$$+\sum_{k=1}^{\infty}D_k\left[\mu_k\operatorname{ch}\mu_k\,\frac{b-a}{R}+h_2R\operatorname{sh}\mu_k\,\frac{b-a}{R}\right]\times J_0\left(\mu_k\,\frac{r}{R}\right)\frac{\operatorname{sh}\mu_k\,\frac{z}{R}}{\operatorname{sh}\mu_k\,\frac{a}{R}}+\frac{2\pi}{a^2}\,(h_2-h_1)\sum_{n=1}^{\infty}(-1)^{n+1}\,nI_0\left(\frac{n\pi}{a}\,r\right)\times$$

$$\times\sin\frac{n\pi}{a}\,z\left[\frac{n\pi}{a}\,I_1\left(\frac{n\pi}{a}\,R\right)+h_1I_0\left(\frac{n\pi}{a}\,R\right)\right]^{-1}\times\sum_{k=1}^{\infty}\frac{D_kJ_0(\mu_k)}{\frac{\mu_k^2}{R^2}+\left(\frac{n\pi}{a}\right)^2}\left[\mu_k\operatorname{ch}\mu_k\,\frac{b-a}{R}+h_2R\operatorname{sh}\mu_k\,\frac{b-a}{R}\right].\quad(6)$$

By fulfilling the continuity condition for the heat flow at the section z = a one obtains

$$\sum_{k=1}^{\infty}D_k\,\frac{\mu_kJ_0\left(\mu_k\,\frac{r}{R}\right)}{R\operatorname{sh}\mu_k\,\frac{a}{R}}\left[\mu_k\operatorname{ch}\mu_k\,\frac{b}{R}+h_2R\operatorname{sh}\mu_k\,\frac{b}{R}\right]=$$

$$=\frac{1}{a}+\frac{2h_1}{\pi}\sum_{n=1}^{\infty}(-1)^n\,I_0\left(\frac{n\pi}{a}\,r\right)\times\left[\frac{n\pi}{a}\,I_1\left(\frac{n\pi}{a}\,R\right)+h_1I_0\left(\frac{n\pi}{a}\,R\right)\right]^{-1}+\frac{2}{a}\,(h_2-h_1)\times$$

$$\times\sum_{n=1}^{\infty}\left(\frac{n\pi}{a}\right)^2I_0\left(\frac{n\pi}{a}\,r\right)\left[\frac{n\pi}{a}\,I_1\left(\frac{n\pi}{a}\,R\right)+h_1I_0\left(\frac{n\pi}{a}\,R\right)\right]^{-1}\times$$

$$\times\sum_{k=1}^{\infty}\frac{D_kJ_0(\mu_k)}{\frac{\mu_k^2}{R^2}+\left(\frac{n\pi}{a}\right)^2}\left[\mu_k\operatorname{ch}\mu_k\,\frac{b-a}{R}+h_2R\operatorname{sh}\mu_k\,\frac{b-a}{R}\right].$$

Multiplying this relation by $rJ_0[\mu_k(r/R)]$, and integrating the result with respect to r from 0 to R and by changing from the unknowns D_k to the unknowns A_k, by means of

$$A_k=\frac{1}{2}\,D_k\,\frac{\mu_k^2J_0(\mu_k)}{\operatorname{sh}\mu_k\,\frac{a}{R}}\times\left[\mu_k\operatorname{ch}\mu_k\,\frac{b}{R}+h_2R\operatorname{sh}\mu_k\,\frac{b}{R}\right]\left(1+\frac{h_2^2R^2}{\mu_k^2}\right),\quad(7)$$

one obtains for the unknowns A_k the infinite system of linear equations

$$A_m=\sum_{k=1}^{\infty}\alpha_{mk}A_k+\beta_m \quad (m=1,\ 2,\ \ldots), \tag{8}$$

where

$$\alpha_{mk}=\frac{4}{a}\ (h_2-h_1)\ B_k\mu_m(\mu_k^2+h_2^2R^2)^{-1}\times\sum_{n=1}^{\infty}\left(\frac{n\pi}{a}\right)^2\left[\frac{n\pi}{a}I_1\left(\frac{n\pi}{a}R\right)+h_2I_0\left(\frac{n\pi}{a}R\right)\right]\times$$

$$\times\left\{\left[\frac{\mu_k^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right]\left[\frac{\mu_m^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right]\times\left[\frac{n\pi}{a}I_1\left(\frac{n\pi}{a}R\right)+h_1I_0\left(\frac{n\pi}{a}R\right)\right]\right\}^{-1} \tag{9}$$

and

$$\beta_m=\frac{h_2R^2}{\mu_m a}+2\mu_m\frac{h_1}{a}\sum_{n=1}^{\infty}(-1)^n\left[\frac{n\pi}{a}I_1\left(\frac{n\pi}{a}R\right)+h_2I_0\left(\frac{n\pi}{a}R\right)\right]\times$$

$$\times\left\{\left[\frac{\mu_m^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right]\left[\frac{n\pi}{a}I_1\left(\frac{n\pi}{a}R\right)+h_1I_0\left(\frac{n\pi}{a}R\right)\right]\right\}^{-1}. \tag{10}$$

In the above the following notation was introduced:

$$B_k=\operatorname{sh}\mu_k\frac{a}{R}\left[\mu_k\operatorname{ch}\mu_k\frac{b-a}{R}+h_2R\operatorname{sh}\mu_k\frac{b-a}{R}\right]\times\left[\mu_k\operatorname{ch}\mu_k\frac{b}{R}+h_2R\operatorname{sh}\mu_k\frac{b}{R}\right]^{-1}. \tag{11}$$

It will now be shown that the infinite system (8) is completely regular.

First of all, by using the obvious inequalities

$$\frac{\operatorname{ch}(y-x)\operatorname{sh}x}{\operatorname{sh}y}\leqslant\frac{1}{2}\ \{1+\exp[-2(y-x)]\}\leqslant 1 \text{ и } \frac{\operatorname{sh}(y-x)\operatorname{sh}x}{\operatorname{sh}y}\leqslant\frac{1}{2}[1-\exp(-2x)]\leqslant\frac{1}{2},\quad y>x\geqslant 0,$$

one can easily show that

$$B_k\leqslant 1 \quad (k=1,\ 2,\ \ldots). \tag{12}$$

Moreover, the function [0, R] which is continuous on the interval $I_0[(n\pi/a)\ r]$ is expanded into a uniformly convergent Dini series [3],

$$I_0\left(\frac{n\pi}{a}r\right)=\sum_{k=1}^{\infty}C_kJ_0\left(\mu_k\frac{r}{R}\right) \quad (0\leqslant r\leqslant R), \tag{13}$$

where μ_k are the roots of the transcendental equation (5). The expansion coefficients C_k are then given by

$$C_k=\frac{2}{R}\left[\frac{n\pi}{a}I_1\left(\frac{n\pi}{a}R\right)+h_2I_0\left(\frac{n\pi}{a}R\right)\right]\times\left\{\left[\frac{\mu_k^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right]J_0(\mu_k)\left(1+\frac{h_2^2R^2}{\mu_k^2}\right)\right\}^{-1}. \tag{14}$$

Multiplying the relation (13) with the coefficients (14) by r and then integrating, this being correct in view of the uniform convergence of the series (13), and setting r = R, one obtains

$$\frac{a}{n\pi}I_1\left(\frac{n\pi}{a}R\right)=2\sum_{k=1}^{\infty}J_1(\mu_k)\left[\frac{n\pi}{a}I_1\left(\frac{n\pi}{a}R\right)+h_2I_0\left(\frac{n\pi}{a}R\right)\right]\times$$

$$\times\left\{\mu_kJ_0(\mu_k)\left[\frac{\mu_k^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right]\left(1+\frac{h_2^2R^2}{\mu_k^2}\right)\right\}^{-1}. \tag{15}$$

If one replaces h_2 by using the expression (5) and by employing (12) and (15), one obtains the estimate

$$\sum_{k=1}^{\infty} |\alpha_{mk}| \leqslant \mu_m \frac{4}{aR}\left(1-\frac{h_1}{h_2}\right) \times \sum_{n=1}^{\infty}\left(\frac{n\pi}{a}\right)^2 \left\{\left[\frac{\mu_m^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right] \times\right.$$

$$\left.\times\left[\frac{n\pi}{a} I_1\left(\frac{n\pi}{a}R\right)+h_1 I_0\left(\frac{n\pi}{a}R\right)\right]\right\}^{-1} \times \sum_{k=1}^{\infty} J_1(\mu_k)\left[\frac{n\pi}{a} I_1\left(\frac{n\pi}{a}R\right)+h_2 I_0\left(\frac{n\pi}{a}R\right)\right] \times$$

$$\times\left\{\mu_k J_0(\mu_k)\left(1+\frac{h_2^2R^2}{\mu_k^2}\right)\left[\frac{\mu_k^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right]\right\}^{-1} = \mu_m \frac{2}{aR}\left(1-\frac{h_1}{h_2}\right)\sum_{n=1}^{\infty}\frac{n\pi}{a} I_1\left(\frac{n\pi}{a}R\right) \times$$

$$\times\left\{\left[\frac{\mu_m^2}{R^2}+\left(\frac{n\pi}{a}\right)^2\right]\left[\frac{n\pi}{a} I_1\left(\frac{n\pi}{a}R\right)+h_1 I_0\left(\frac{n\pi}{a}R\right)\right]\right\}^{-1} \leqslant$$

$$\leqslant \mu_m \frac{2}{aR}\left(1-\frac{h_1}{h_2}\right)\sum_{n=1}^{\infty}\frac{1}{\left(\frac{n\pi}{a}\right)^2+\frac{\mu_m^2}{R^2}} \leqslant 1-\frac{h_1}{h_2}. \qquad (16)$$

In the above it was assumed that [4]

$$\sum_{k=1}^{\infty}\frac{1}{k^2+x^2} = \frac{\pi}{2x}\left[\operatorname{cth}\pi x - \frac{1}{\pi x}\right] \qquad (17)$$

as well as the obvious inequality

$$\operatorname{cth} y - \frac{1}{y} \leqslant 1 \text{ for } 0 \leqslant y < \infty. \qquad (18)$$

It will now be shown that the free terms of the infinite system (8) are bounded. By rewriting (10) as

$$\beta_m = \frac{h_2R^2}{\mu_m a} + 2\mu_m \frac{h_1}{a}\sum_{n=1}^{\infty}(-1)^n\left[\left(\frac{n\pi}{a}\right)^2+\frac{\mu_m^2}{R^2}\right]^{-1} +$$

$$+2\mu_m \frac{h_1}{a}(h_2-h_1)\sum_{n=1}^{\infty}(-1)^n I_0\left(\frac{n\pi}{a}R\right) \times$$

$$\times\left\{\left[\left(\frac{n\pi}{a}\right)^2+\frac{\mu_m^2}{R^2}\right]\left[\frac{n\pi}{a} I_1\left(\frac{n\pi}{a}R\right)+h_1 I_0\left(\frac{n\pi}{a}R\right)\right]\right\}^{-1} \qquad (19)$$

and using the formula [4]

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2+x^2} = \frac{1}{2x}\left(\frac{\pi}{\operatorname{sh}\pi x} - \frac{1}{x}\right), \qquad (20)$$

it can be shown that

$$\beta_m \leqslant R h_2 \operatorname{sh}^{-1}\mu_m \frac{a}{R} < \infty \text{ и } \beta_m \to 0 \text{ for } m \to \infty. \qquad (21)$$

It follows from the estimates (16) and (21) that for $h_1 \leq h_2$ the infinite system (8) is completely regular and its solution is bounded. In this case, as one knows from the theory of completely regular infinite systems of linear equations (5), the unknowns $A_m (m = 1, 2, \ldots)$ can be determined with any required accuracy. Proceeding from A_k to D_k by means of the formula (7), and substituting the obtained values of D_k into (4) and (6), one can evaluate the temperature with any required accuracy at any point of the cylinder. In view of the fact that one has selected the domains (I) and (II) arbitrarily, one can assume without loss of generality that one has always $h_1 \leq h_2$, since otherwise, having selected the origin of the coordinate system at the top base of the cylinder, the z axis could be directed downwards and as a result the domains (I) and (II) would merely exchange their position. Employing similar though somewhat more difficult considerations one can again obtain a completely regular system as in (8) with an estimate as in (16).

The case $h_1 = 0$ is now considered. The formulas (9)-(10) now simplify to

$$\alpha_{mk} = \frac{4}{aR} h_2 \mu_m B_k \left(1 + \frac{h_2^2 R^2}{\mu_k^2}\right)^{-1} \times$$

$$\times \sum_{n=1}^{\infty} \frac{n\pi}{a} \left[\frac{n\pi}{a} I_1\left(\frac{n\pi}{a} R\right) + h_2 I_0\left(\frac{n\pi}{a} R\right)\right] \times$$

$$\times \left\{I_1\left(\frac{n\pi}{a} R\right)\left[\frac{\mu_k^2}{R^2} + \left(\frac{n\pi}{a}\right)^2\right]\left[\frac{\mu_m^2}{R^2} + \left(\frac{n\pi}{a}\right)^2\right]\right\}^{-1} > 0 \tag{22}$$

and

$$\beta_m = R^2 h_2 / a\mu_m > 0. \tag{23}$$

Employing (5), (12), (15), and (18) one can obtain an estimate similar to (16):

$$\sum_{k=1}^{\infty} |\alpha_{mk}| \leqslant 1 - \rho_m \qquad (m = 1, 2, \ldots), \tag{24}$$

where

$$\rho_m = 1 - \operatorname{cth}(\mu_m a/R) + (R/\mu_m a). \tag{25}$$

Moreover, the free terms of the system (8) satisfy in this case the condition

$$\beta_m \leqslant \rho_m K, \tag{26}$$

where

$$K = Rh_2\left[1 + \frac{a\mu_1}{R}\left(1 - \operatorname{cth}\frac{a\mu_1}{R}\right)\right]^{-1}, \tag{27}$$

and μ_1 is the smallest positive root of Eq. (5).

It is known [5] that if the estimates (24) and (26) are satisfied the system (8) is regular and possesses a bounded solution $|A_m| \leq K$ which can be found by using, say, the method of successive approximations.

Moreover, since the coefficients (22) and the free terms (23) of the system (8) are positive it can be asserted that the system has a unique positive solution which approaches zero; this is its principal solution and may be found by the reduction method.

Since the system (8) is regular in the case of $h_1 = 0$, the following should be noted. By definition [5] one has

$$\rho_m = 1 - \sum_{k=1}^{\infty} |\alpha_{mk}| \qquad (m = 1, 2, \ldots). \tag{28}$$

By setting $B_k = 1$, one increases $\sum_{k=1}^{\infty}|\alpha_{mk}|$, and thus decreases ρ_m, in view of the estimate (26). If now one uses (12), then ρ_m is greater for any fixed m, and K is smaller, which does not change, of course, the estimate (26) for a given m.

The proposed procedure can easily be extended also to the case in which the cylinders (I) and (II) are made of different materials; then the continuity condition of the heat flow at the section $z = a$ must be replaced by

$$\left.\frac{\partial\theta_1}{\partial z}\right|_{z=a} = \frac{\lambda_2}{\lambda_1}\left.\frac{\partial\theta_2}{\partial z}\right|_{z=a}. \quad (29)$$

The infinite system thus obtained proves again to be completely regular. Of course, there may be more than two domains in which the heat-exchange coefficient remains constant on the lateral surface of the cylinder.

NOTATION

λ_i, coefficient of thermal conductivity for the i-th portion; h_i, heat-exchange coefficient of the i-th portion; $\theta = (T - T_{med})/(T_0 - T_{med})$, dimensionless temperature; $J_0(x)$, $J_1(x)$, Bessel functions of the first kind; $I_0(x)$, $I_1(x)$, modified Bessel functions. Indices: med, surrounding medium; 1) domain (I); 2) domain (II).

LITERATURE CITED

1. N. Kh. Arutyunyan and B. L. Abramyan, Torsion of Elastic Bodies [in Russian], Fizmatgiz, Moscow (1963).
2. G. A. Grinberg, Collected Problems of Mathematical Theory of Electrical and Magnetic Phenomena [in Russian], Izd. Akad. Nauk SSSR, Moscow—Leningrad (1948).
3. N. N. Lebedev, Special Functions and Their Applications [in Russian], Fizmatgiz (1963).
4. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products Academic Press (1966).
5. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis [in Russian], Fizmatgiz, Moscow—Leningrad (1962).